MATHEMATICAL THEORY OF THINK-A-DOT

BENJAMIN L. SCHWARTZ, Institute for Defense Analyses

1. Introduction. *Think-a-Dot* is an inexpensive mathematical toy recently put on the market. Its manufacturer claims (with considerable justice, in this author's opinion) that it illustrates many principles of digital computers. Its design also raises a number of interesting mathematical questions in its own right. Some of those are discussed in this article.

The results given here are new, since *Think-a-Dot* is new. The writer believes that they are neither trivial nor obvious. Yet the mathematical foundation required to obtain them is so minimal that this paper might have been written by a high school student! All that is required is to know that the sum of two integers is even if and only if either both summands are even, or both are odd. It is the author's hope that because of this very bare set of prerequisites, this paper may prove understandable to readers down to the junior high school level. To this end, we have tried to avoid unnecessary advanced mathematical terminology. We thereby illustrate that mathematical research can go on at almost any level.

2. Description of Think-a-Dot. *Think-a-Dot* is a box, at the top of which are three holes into any one of which a marble may be inserted. Figure 1 gives a general view of the device. When a marble is inserted, it moves under gravity power down through a set of inclined planes, and emerges on a track at the bottom. Where two planes meet internally there is a "switch" which directs the marble to either the right or left hand emerging path from the switch. There is also a switch at each entry port at the top. Each time a marble goes through a switch, it reverses the switch setting, so that the next marble will go the other way. This is the only way switches may be changed after an initial pattern is set up. On a vertical face of the toy, the states of all switches are displayed through color-coded openings each of which shows either a yellow or blue dot. As marbles are successively dropped through, the color pattern of the dots changes.

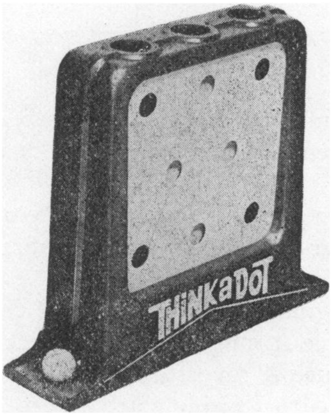

FIG. 1

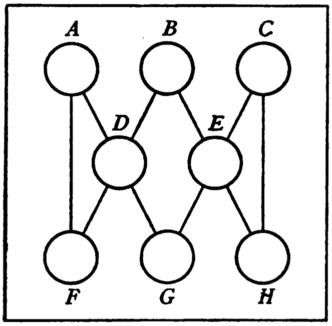

FIG. 2

3. Example. Figure 2 shows the vertical face of *Think-a-Dot* with the eight switches and the available paths between them. Suppose all switches were originally set to the left-hand branches. If a marble were dropped into the center hole, it would pass through switch B and emerge at the left, continue on down to switch D on the inclined plane schematically shown; again emerge on the left, and finally go on to F, where it would at last come out onto the output track from the left hand side of F. In following this path, it would cause the three switches traversed, B, D, and F to reverse their settings. If, then, a second marble were dropped into the center hole, it would now emerge on the right of B (since that switch had been reversed by the first marble) and go to switch E. This switch would still be set for a left hand branch, so the marble would leave it going to the left and pass through G, and come out onto the track. After these two operations, switch B would have been reversed twice, and hence would be in its original state. Switches D, E, F, and G would each have been actuated once, and hence be reversed. And switches A, C, and H would not have been changed at all. The new dot pattern would differ from the original in positions D, E, F, and G.

4. First problem. The manufacturer suggests that an initial dot pattern be set up by tilting the device to one side or the other. The user can then select an arbitrary new pattern, and try to attain it by successive marble drops. (Hence the name.)

The question immediately arises, is every pattern obtainable from every other? Surprisingly, the answer to this is "no," as we shall now show. This means that a user following the manufacturer's instructions can very well find himself engaged in a task that is impossible of fulfillment!

In proving this, it is first convenient to define a new term. We shall say two patterns of switch settings are in *near-agreement* if they agree on all switch positions except the lower center, switch G. This is not the same thing as saying that they agree except for (any) one switch. The term "near-agreement" of two patterns means that they differ specifically on G and on no other switch.

We can now prove:

THEOREM 1. *If any initial pattern is given, then the pattern in near-agreement with it cannot be obtained by marble drops.*

Proof. We shall use the so-called indirect method in which we assume temporarily that the result can be achieved and then show that this leads to a contradiction.

Suppose, therefore, that a pattern is given and there is a sequence of marble drops that converts it to the pattern in near-agreement. Consider first switch A. This is actuated only by marble drops into the left hand hole in the top. Since this switch must be returned to its original state at the end of the operation, the number of marbles dropped into hole A must be even, say $2k$ where k is an integer to be found (if possible). Because A reverses setting at each passage of a marble, we can see that half the time, the exiting marble from A will go to D, and half the time to F. Hence D and F will each be actuated just k times as a result of drops into the left hand hole. These latter two switches can also be actuated by other marbles, a point we shall take care of shortly.

A similar argument shows that the number of drops in the center must be even, say $2m$; and also the number of drops on the right, say $2n$.

Returning to D, we observe that its total number of actuations from either A or B will be $(k+m)$. But again, this must be even, since D is returned to its original state at the end. Let us say

$$k + m = 2s. \tag{1}$$

Similarly, analyzing E, we conclude that

$$m + n = 2t. \tag{2}$$

Looking now at the lower three switches, we have at F just k inputs from A, and s from D.

Since F is not to be changed,

$$(k + s) \text{ is even.} \tag{3}$$

Arguing similarly at H, we find

$$(n + t) \text{ is even.} \tag{4}$$

And finally since G *is* to change, we observe that

$$(s + t) \text{ is odd.} \tag{5}$$

Now there are two possibilities. Either k is even or it is odd.

(6) First suppose k is even.

(7) Then from (1) and (6), m is even

(8) From (3) and (6), s is even

(9) From (2) and (7), n is even

(10) From (4) and (9), t is even

But from (8) and (10), $(s+t)$ is even, contradicting (5). Hence k could not be even.

The reader can quickly verify in like manner that the initial assumption that k is odd leads to the conclusion that s and t are both odd, which again violates (5). Hence the required numbers k, m, n cannot be found, and the transformation desired is impossible. This proves the theorem.

5. Mathematical structure of the patterns. Faced with this disappointing result, we now naturally turn to such questions as the following: Under what conditions can a given pattern be obtained from another? When the transformation is possible, find an algorithm (that is, a systematic procedure) to achieve it.

In order to answer problems of this sort, we now build up a somewhat more elaborate mathematical machinery of the dot patterns.

LEMMA 1. *Any desired pattern of the three switches A, D, F can be obtained from any initial pattern, with between 0 and 7 drops in the left hand hole (and no drops in the other holes).*

Proof. We shall show that as marbles are dropped in the left hand hole, the three switches progress in sequence through the eight different patterns they can take. This being true, it does not matter where we start in showing it. Suppose then that the switches are initially all set to the left. On succeeding passes, it can be verified directly that the patterns assumed will be as tabulated below.

Switch \ Iteration	0	1	2	3	4	5	6	7	8
A	L	R	L	R	L	R	L	R	L
D	L	L	R	R	L	L	R	R	L
F	L	R	L	R	R	L	R	L	L

For example, on iteration 3, the marble will enter A, which is set to the right. It will leave A on the right hand plane, leading to D. At D, it will be directed to the right since D is shown set to the right on this iteration. It will therefore go on to G, and drop onto the track from there. In this path, it goes through A, D, and G which are therefore reversed at the next iteration, while the other switches are unchanged. Since the table includes only A, D, and F, we find, at iteration 4, that A and D have changed, and F has not.

By this laborious but straightforward process, it can be shown that the table is correct. On examination of it, we can see that the original pattern reappears on iteration 8, and at intermediate steps, all other possible patterns of the designated three switches occur. This completes the proof.

LEMMA 2. *Any desired pattern of the three switches C, E, H, can be obtained from any initial pattern, with between* 0 *and* 7 *drops in the right hand hole.*

This is a symmetrical statement and the proof is similar to Lemma 1.

THEOREM 2. *Any pattern* (*of all eight switches*) *can be brought into agreement or near-agreement with any other pattern with* 15 *marble drops or less.*

Proof. First, with zero or one drop in the center hole, obtain conformity of switch B to the desired pattern. *Next,* by Lemma 1, no more than 7 drops in the left hand hole are needed to obtain agreement in switches A, D, and F. And these drops cannot disturb B. *Finally,* by Lemma 2, with no more than 7 drops in the right hand hole, attain agreement in switches C, E, and H. These do not disturb any previously obtained agreements.

So all switches except possibly G can be brought to agree with those of the desired pattern. And this is the theorem.

THEOREM 3. *If one pattern can be obtained from a second, then the second can be obtained from the first.*

Proof. Let it be possible to obtain pattern Γ from pattern Δ. Begin with Δ. Proceed to obtain Γ by marble drops. Using Theorem 2, continue to make drops in order to transform Γ back to either Δ or the pattern in near-agreement with Δ. If it were actually near-agreement, then we would have produced overall a sequence of drops that transformed Δ into a pattern in near-agreement with itself. This is impossible by Theorem 1. Hence the result must actually be Δ.

So the second half of the construction actually did transform Γ into Δ, as required.

6. Equivalent classes of patterns. Theorem 3 shows that all patterns can be separated into distinct, nonoverlapping classes in such a way that any two patterns in the same class can be obtained from one another; and no pattern in another class can be. These classes are termed, in the language of mathematicians, equivalence classes.

We have shown that there are at least two such classes, since a pattern and the one in near-agreement with it are not in the same class, by Theorem 1. We shall now show that there are no more than two classes.

THEOREM 4. *There are only two equivalence classes.*

Proof. Consider any pattern and the one in near-agreement with it. These are in different classes. Now take any third pattern. We wish to show it is in the same class as one of the two already chosen. By Theorem 2, the third pattern can be transformed into one of the first two. And this proves the theorem.

7. Some particular pattern pairs. Our next question will be to determine the conditions under which two patterns actually are in the same equivalence class. For this, we shall require several intermediate results.

LEMMA 3. *If two patterns agree in all except position F, they are in different classes.*

The proof of this follows the lines of Theorem 1. The reader can easily provide the details. By symmetry, the same result holds for switch H.

LEMMA 4. *If two patterns agree in all except position A, they are in different classes.*

This proof is a bit more complicated. Arguing as in Theorem 1, and assuming the two patterns are such that one could be gotten from the other, we specify the number of drops in the left, center and right hand holes as $2k+1$, $2m$, and $2n$. This is so, since A is to be reversed at the end of the operation and hence, must have been through an odd number of changes, while B and C are to be in their original states. Now we must distinguish two cases accordingly as A was originally set left or right.

If it was initially set to the left, then at D and E we have respectively $(k+m)$ and $(m+n)$ actuations. Both these numbers must be even, say $2s$ and $2t$ respectively. And at F, G, and H, we have $[(k+1)+s]$, $(s+t)$, and $(t+n)$ respectively, all of which must be even. It can be shown in familiar fashion as in Theorem 1 that this is impossible, regardless of whether k is assumed even or odd.

On the other hand, if A was originally set to the right, then at D and E, the inputs are $[(k+1)+m]$ and $(m+n)$, which we again set equal to $2s$ and $2t$. The three lower switches are actuated $(k+s)$, $(s+t)$, and $(t+n)$ times respectively, all of which are to be even numbers. Again, the impossibility of this set of conditions follows from either the assumption that k is even or k is odd. Details are left to the reader. The lemma now follows, and applies by symmetry also to switch C.

The reader will also easily be able to provide the proof of the following similar result.

LEMMA 5. *If two patterns agree in all except position B, they are in different classes.*

8. Other pairs of patterns. It remains to consider the switches in the center line, D and E. Here, for a change, we have a different conclusion.

LEMMA 6. *If two patterns agree in all positions except D, they are in the same equivalence class.*

As with Lemma 4, there are two cases to consider. These are similar (but not identical). We shall go through one in detail, leaving the other for the reader. Suppose D is initially set to the left.

The three inputs at the top are as usual $2k$, $2m$, and $2n$. The inputs at D and E are therefore $(k+m)$ and $(m+n)$, which must be odd and even, respectively, say.

(11) $$k + m = 2s + 1,$$

(12) $$m + n = 2t.$$

Since D was initially set left, the three inputs to the bottom switches are

(13) at F, $k + (s + 1)$, even;

(14) at G, $s + t$, even;

(15) at H, $t + n$, even;

(16) Now if k is odd, then

(17) from (11) and (16), m is even;

(18) from (12) and (17), n is even;

(19) from (15) and (18), t is even;

(20) from (14) and (19), s is even.

And (16) and (20) are consistent with (13). Hence $k=1$, $m=n=0$ is a candidate solution. And in fact, since it satisfied (11)–(15), it must meet the required conditions. Therefore an acceptable sequence of marble drops is simply: drop two in the left hand hole. It is immediately clear that this does, in fact, just reverse D. One of the two goes left and reverses A and F, the other goes right to D, then left to F and therefore reverses all three, A, D, and F.

The reader should verify that similar solutions can be found if D is initially set to the right. Having done so, he will have completed the proof of the lemma.

By symmetry, the result of Lemma 6 also applies to switch E.

9. A test for equivalence. An empirical way to test two given patterns to see if one can be obtained from the other is to follow the construction of Theorem 2 to bring them into agreement or near-agreement.

However, for our final theoretical result, we give an easier test involving no experimentation. We first define the *test set* as the set of six switches: A, B, C, F, G, and H.

LEMMA 7. *Two patterns are in the same equivalence class if they agree on the test set. They are in different classes if they differ on exactly one switch in the test set.*

This is essentially another way of saying Theorem 1 and Lemmas 3–6 somewhat more compactly.

LEMMA 8. *Two patterns are in the same equivalence class if the number of switches in the test set on which they disagree is* 2.

Proof. Consider two patterns Δ and Γ that differ on exactly two switches of the test set, say A and B. Let Λ be a pattern that agrees with Γ on A, with Δ on B, and with both elsewhere on the test set. Then Δ is not in the same equivalence class as Λ, by Lemma 7, and likewise Γ is not in the same class as Λ. But there is only one class that Λ does not belong to (since there are only two classes altogether); and we have shown that both Γ and Δ belong to it. The lemma follows.

This leads to:

THEOREM 5. *Two patterns are in the same equivalence class if and only if the number of switches in the test set on which they agree is even* (*namely* 0, 2, 4 *or* 6).

Proof. The case of differing on no switches is Lemma 6. The case of differing on 2 switches is Lemma 8. So now, suppose Γ and Δ differ on four of the test set switches, say A, B, C, and F. Then choose Λ to agree with Δ on A and B; agree with Γ on C and F; and agree with both elsewhere on the test set. Then by Lemma 8, Δ and Λ are in the same class; and Γ and Λ are in the same class. Hence Γ and Δ are in the same class, as required.

This proves the case in which the number of nonmatching switches is 4. The final case, when that number is 6, is handled similarly. An auxiliary pattern is constructed agreeing with one of the given pair on two switches of the test set, and the other on the remaining four. Then the given patterns are both known to be in the same class as the newly constructed one, hence, the same as each other. This completes the proof.

10. Concluding observations. We can recapitulate the theory briefly. All of the possible dot patterns of the eight switches (there are 256 of them) fall into two nonoverlapping classes. In each class, any pattern can be obtained from any other. One way to do this in 15 marble drops or less is described in Theorem 2. If two patterns are in different classes, neither can be obtained from the other. To determine whether two given patterns are in the same class, it suffices to count modulo 2 the number of positions on which they agree in the so-called test set, consisting of the top three and bottom three switches (omitting the middle two).

The richness of the mathematical theory that has been built on so modest a foundation comes as a surprise to the author and may also be unexpected for some readers. There remain many further extensions and unsolved problems. One of the latter of particular interest to puzzlers would be to develop an algorithm for the shortest program (i.e., minimum number of drops) to transform one pattern to another.